Math 170b - Intro to numerical analysis: Approximation & non-linear equations.

Topics: Numerical methods for solving non-linear equations and for approximation.

Instructor: Prof Rayan Saab (rsaab@ucsd.edu)
Office hours: Tuesday 2:00 - 4:00 pm (or by appointment)

TA: Shi (Fox) Cheng

Text: Kincaid & Cheney, Numerical Analysis: Mathematics of Scientific Computing, 3rd edition

Course webpage: math.ucsd.edu/~rsaab → teaching → click on course
↖ check regularly for HW & announcements

Homework: 5 assignments (essentially bi-weekly) 20% of grade

Midterms: 2 exams (20% each) weeks 4 & 8
Final: March (40% of grade)

Alternative grading scheme (HW: 20%, Higher midterm 20%, Final 60%)

Lecture 1: · Review of Calculus (Sec 1.1 in book)

Limits: $\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$
s.t. $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$

If there is no such L, then the limit does not exist at c.

example: $\lim_{x \to 2} x^2 = 4$

proof (informal) Want $|x^2 - 4| < \varepsilon$ whenever $0 < |x-2| < \delta(\varepsilon)$

$-\delta < x - 2 < \delta$

but then $|x^2 - 4| = |x-2||x+2| < \underbrace{\delta(4+\delta)}$

$\delta^2 + 4\delta = \varepsilon$ so $\delta = -2 + \sqrt{4+\varepsilon}$

formal proof: $\forall \varepsilon > 0$ let $\delta = -2 + \sqrt{4+\varepsilon}$

If $|x-2| < \delta$

then

$|x^2 - 4| = |x-2||x+2| < \delta \cdot (\delta + 4) = (-2 + \sqrt{4+\varepsilon})(2 + \sqrt{4+\varepsilon}) = \varepsilon.$

example 2: $f(x) = \frac{|x|}{x} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$ (undefined at 0)

$\lim_{x\to 0} \frac{|x|}{x} = ?$

Suppose $\lim_{x\to 0} \frac{|x|}{x} = L$ then $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\left|\frac{|x|}{x} - L\right| < \varepsilon$ when $|x| < \delta$

take $\varepsilon = 1$

then $\left|\frac{|x|}{x} - L\right| < 1$ when $|x| < \delta$

but choosing $x = \delta/2$: $|1 - L| < 1 \iff 0 < L < 2$

choosing $x = -\delta/2$: $|-1 - L| < 1 \iff -2 < L < 0$ ← contradiction

So no such L exists!

Continuity: f is cont's at c if $\lim_{x\to c} f(x) = f(c)$

e.g. $f(x) = x^2$ is cont. at $x = 2$

$f(x) = \frac{|x|}{x}$ is cont. at $x = 2$ (why?)

$f(x) = \frac{|x|}{x}$ is not cont. at $x = 0$ (why?)

Intermediate Value Theorem: Let $[a, b]$ be an interval & let f be a cont. fct on $[a, b]$. Then for every $y \in [f(a), f(b)]$ (or $[f(b), f(a)]$), there exists $x^* \in [a, b]$ s.t. $f(x^*) = y$.

In words: f assumes all values between $f(a)$ & $f(b)$

Derivative: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If $f'(c)$ exists we say f is differentiable at c.
If f is differentiable at $c \Rightarrow f$ is cont's at c.

Proof:

$$\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = f'(c) \cdot \lim_{x \to c} (x - c)$$

$$= 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c) \Rightarrow \text{cont.}$$

Warning: not all cont's fcts are differentiable!

Notation: $C(\mathbb{R})$ set of all cont's functions on $\mathbb{R}$
$C'(\mathbb{R})$ set of all fcts on $\mathbb{R}$ with f' cont's.

$C'(\mathbb{R}) \subset C(\mathbb{R})$

$C^n(\mathbb{R})$ set of fcts on $\mathbb{R}$ with $f^{(n)}$ continuous
$C^\infty(\mathbb{R})$ set of fcts with all derivatives cont.

$C^\infty(\mathbb{R}) \subset \cdots \subset C^2(\mathbb{R}) \subset C'(\mathbb{R}) \subset C(\mathbb{R})$

e.g. $\sin(x), \cos(x), e^x \in C^\infty(\mathbb{R})$

$C^n([a,b])$ = set of f'ts on $[a,b]$ for which $f^{(n)}$ exists and $f^{(n)}$ is cont.

Taylor's theorem

Version I (Lagrange remainder)

If $f \in C^n[a,b]$ & $f^{(n+1)}$ exists on (a,b)

then $\forall x, c \in [a,b]$, there is a pt. ξ between x & c s.t.

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + \frac{f^{(n+1)}(\xi)(x-c)^{k+1}}{(n+1)!}$$

$E_n(x)$: error term

Example: $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot 1 \cdot x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\forall x < \infty$

exercise: Taylor series for $\sin(x)$, $\cos(x)$, $\ln(x)$. ← $x \in (0,\infty)$

Example: What is the approx. error in computing e^3 with n-terms of the taylor series around 0 (i.e. $c=0$)

By Taylor's theorem with $c=0$

$$e^x = \sum_{k=0}^{m-1} \frac{x^k}{k!} + \underbrace{\frac{e^\xi x^m}{m!}}_{E_{m-1}(x)}$$

$$E_{m-1}(3) = \frac{e^\xi 3^m}{m!} < \frac{e^3 \cdot 3^m}{m!}$$

so $E_0(3) < e^3$

$E_1(3) < e^3 \cdot 3$

$E_2(3) < e^3 \cdot 3^2/2$

$E_3(3) < e^3 \cdot 3^3/3!$

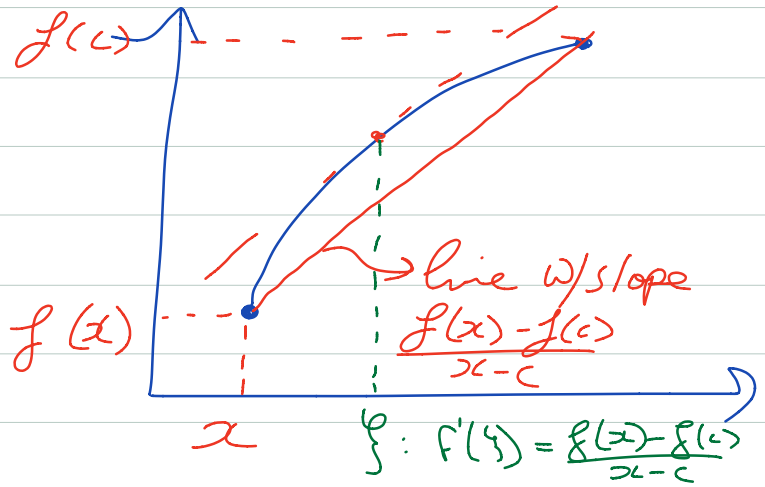
$E_{10}(3) < \frac{e^3 \cdot 3^{10}}{10!} = e^3 \cdot \frac{3 \cdot 3 \cdot \dots \cdot 3}{1 \cdot 2 \cdot \dots \cdot 10}$

$E_{100}(3) < e^3 \cdot \frac{3^{100}}{100!}$ (very small)

Mean Value Theorem: Let $f \in C[a,b]$ & suppose f' exists on (a,b) then $\forall x, c \in [a,b]$

$$f(x) = f(c) + f'(\xi)(x-c)$$ for some ξ between x & c

Alt. statement: $\exists \xi : f'(\xi) = \frac{f(x)-f(c)}{x-c}$



Alternate form of Taylor's theorem:

If $f \in C^{n+1}([a,b])$ & $x, x+h \in [a,b]$

then $f(x+h) = \sum_{k=0}^{n} \frac{f^{k}(x) h^{k}}{k!} + \underbrace{\frac{f^{(n+1)}(\xi) h^{n+1}}{(n+1)!}}_{E_n(h)}$

Taylor's Theorem in 2-variables:

Let $f \in C^{n+1}([a,b] \times [c,d])$ & let (x,y) and $(x+h, y+k)$ both be in $[a,b] \times [c,d] \subseteq \mathbb{R}^2$ then

$$f(x+h, y+k) = \sum_{i=0}^{n} \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^{i} f(x,y) + E_n(h,k)$$

where $E_n(h,k) = \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^{n+1} f(x+\theta h, y+\theta k)$

where $\theta \in [0,1]$.

what is $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^i$?

Think of it as an operator which acts on functions to produce other functions.

Specifically $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^0 f(x,y) = f(x,y)$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^1 f(x,y) = \left(h\frac{\partial f}{\partial x}+k\frac{\partial}{\partial y}f\right)(x,y)$$
$$= hf_x + kf_y$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\frac{\partial^2 f}{\partial x^2}+k^2\frac{\partial^2 f}{\partial y^2}+2hk\frac{\partial^2 f}{\partial x\partial y}$$

...

So $L(x+h,y+k) = \left(f+(hf_x+kf_y)+(h^2f_{xx}+2hkf_{xy}+k^2f_{yy})\right.$
$\left.+\cdots\right)(x,y)$

↑ polynomial in h,k

Example: Write the 2nd order taylor series of $f(x,y)=e^{xy}$
i.e., find $f(x+h,y+k)$

Soln:

we'll need $f(x,y)$ ✓

$$f_x = \frac{\partial f}{\partial x} = ye^{xy} \qquad f_y = xe^{xy}$$

$$f_{xx} = y^2e^{xy},\ f_{yy} = x^2e^{xy},\ f_{xy} = e^{xy}+xye^{xy}$$

So $f(x+h, y+k) = E_2(h,k) + e^{xy} + (hye^{xy} + kxe^{xy})$

$$+\frac{1}{2}\left(h^2y^2e^{xy} + k^2x^2e^{xy} + 2hk\left(e^{xy} + xye^{xy}\right)\right)$$

Example For the same f write $E_1(h,k)$

$$E_1(h,k) = \frac{1}{2!}\left[(\theta h)^2 \bar{y}^2 e^{\bar{x}\bar{y}} + (\theta k)^2 \bar{x}^2 e^{\bar{x}\bar{y}} + 2\theta^2 hk\left(e^{\bar{x}\bar{y}} + \bar{x}\bar{y}e^{\bar{x}\bar{y}}\right)\right]$$

where $\bar{x} = x + \theta h$ & $\bar{y} = y + \theta h$ for some $\theta \in [0,1]$

HW: Sec 1.1 Problems 5, 10, 26

Sec 1.2 Orders of Convergence

For a sequence x_n, we write

$$\lim_{n\to\infty} x_n = L$$

if $\forall \varepsilon > 0$, $\exists r \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon$ whenever $n > r$

e.g. $\lim_{n\to\infty} \frac{n+1}{n} = 1$ bec. $\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon$

$\forall n > r(\varepsilon) := \frac{1}{\varepsilon}$.

Orders of convergence: Suppose $\lim x_n = L$, then we say

$n \to \infty$

the order of conv. of x_n to L is

- Linear if $|x_{n+1} - L| \leq c|x_n - L| \quad \forall n \geq N$ ($c < 1$)

- Superlinear: if $|x_{n+1} - L| \leq \varepsilon_n |x_n - L| \quad \forall n \geq N$ (ε_n sequence with $\varepsilon_n \to 0$)

- Quadratic: if $|x_{n+1} - L| \leq C|x_n - L|^2 \quad \forall n \geq N$ (C const. (not necessarily < 1))

- of order α if $|x_{n+1} - L| \leq C|x_n - L|^\alpha \quad \forall n \geq N$.

Big "Oh" & Little "oh":

- $x_n = O(\alpha_n)$ if $\exists C > 0$ & $N \in \mathbb{N}$ s.t. $|x_n| \leq C|\alpha_n| \quad \forall n \geq N$

- $x_n = o(\alpha_n)$ if $\exists \varepsilon_n \to 0$, $\varepsilon_n \geq 0$ s.t. $|x_n| \leq \varepsilon_n |\alpha_n| \quad \forall n \geq N$

(Intuitively $\lim_{n\to\infty} \frac{x_n}{\alpha_n} = 0$) or ($\alpha_n$ dominates x_n asymptotically)

examples: $e^x - \sum_{k=0}^{n-1} \frac{1}{k!} x^k = O(\frac{1}{n!})$ (why?) (fast convergence)

$\ln 2 - \sum_{k=1}^{n-1} (-1)^{k-1} \frac{1}{k} = O(\frac{1}{n})$ (why?) (slow convergence)

Similar notation for functions (See book!)

- $f(x) = O(g(x))$ if $\exists c, r$ s.t. $f(x) \leq C g(x)$ $\forall x \geq r$
- $f(x) = o(g(x))$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

Mean Value Theorem for Integrals

Let $u \& v \in C([a,b])$ & suppose $v \geq 0$ (i.e. $v(x) \geq 0 \ \forall x \in [a,b]$). Then, $\exists \xi \in [a,b]$ with

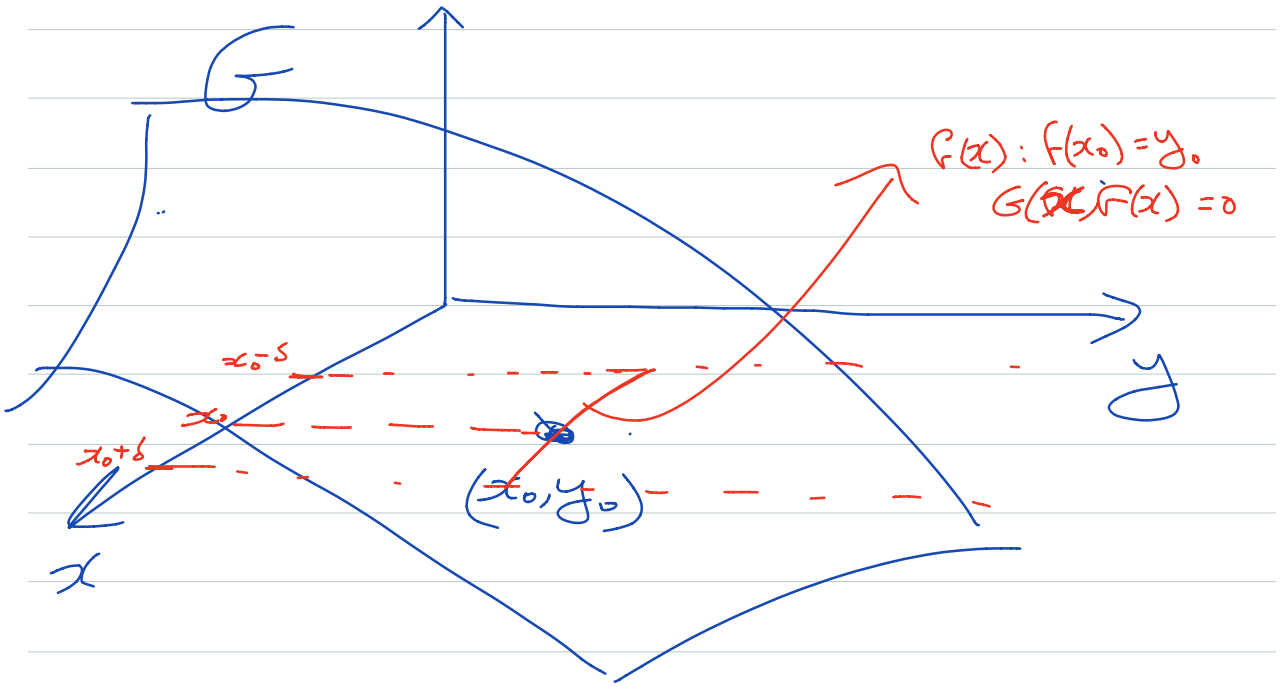
$$\int_a^b v(x)u(x)\,dx = u(\xi)\int_a^b v(x)\,dx$$

Proof: See book (exercise)

Also, see book for def'n of supremum, infimum.

Implicit Fct Theorem:

Let G be a fct of two real variables that is continuously differentiable in a neighborhood of (x_0, y_0). If $G(x_0, y_0) = 0$ & $\frac{\partial G}{\partial y} \neq 0$ at (x_0, y_0) then there is a $\delta > 0$ & f continuously differentiable for $|x - x_0| < \delta$ s.t. $f(x_0) = y_0$ & $G(x, f(x)) = 0 \ \forall x : |x - x_0| < \delta$.



Example: $G(x,y) = x^7 + 2y^8 - y^3$

Note: $G(x,y)$ is cont'sly differentiable

$$\frac{\partial G}{\partial y} = 16y^7 - 3y^2$$

- $G(0,0) = 0$ but $\frac{\partial G}{\partial y}\Big|_{(0,0)} = 0$ so IFT does not apply
- $G(-1,-1) = 0$ & $\frac{\partial G}{\partial y}\Big|_{(-1,-1)} = 13$ so IFT applies!

so $\exists f$ (continuously differentiable): $f(-1) = -1$ & $G(x, f(x)) = 0$ in the neighborhood of $x_0 = -1$.

Example: Let y be defined implicitly by the equation

$$x^3 - y^7 + 4x^2 + y^4 - 24 = 0$$

Find $\frac{dy}{dx}\Big|_{(2,1)}$

Differentiate: (recall that y is a function of x by IFT.)

$$3x^2 - 7y^6\frac{dy}{dx} + 8x + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = (7y^6 - 4y^3)^{-1} \cdot (3x^2 + 8x)$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(2,1)} = \frac{1}{3} \cdot 28.$$

HW: Sec 1.2, #7, 15, 22, 33, 35

Chapter 3: Solutions of non-linear equations

Goal: Want to find roots of equations (zeros of functions)

i.e.,

Find x so that $f(x) = 0$

$x \in \mathbb{R}$ or $x \in \mathbb{R}^n$

f: $\mathbb{R} \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}^m$

Next few lectures: Bisection method, Newton's method, Secant method

Section 3.1: Bisection Method

Based on simple observations

- If $f(a) > 0$ & $f(b) < 0$ (or vice versa) — either way $f(a)f(b) < 0$ — & if f is continuous on $[a,b]$ then (by the Intermediate value theorem) there is some point (say x) in $[a,b]$ s.t. $f(x) = 0$.
- If $c = \frac{a+b}{2}$ (midpt of the interval) then x is either in $[a,c]$ or $(c,b]$. So we check: if $f(c)f(a) < 0$ then $\exists x \in [a,c]$ with $f(x) = 0$

otherwise $f(c)f(b)<0$ & $\exists x \in (c,b]$ with $f(x)=0$ (or $f(c)=0$)

- We can keep going this way until $|f(c)|<\varepsilon$ for some tolerance ε.

Example:

Use the bisection method to find a root of $e^x = x^2$.

Sol'n: Let $f(x) = e^x - x^2$ & note that

- $f(-1) = \frac{1}{e} - 1 < 0$ & $f(0) = e^0 - 1 > 0$

so we can take $[a_1, b_1] = [-1, 0]$

$c_1 = \frac{-1+0}{2} = -\frac{1}{2}$, $f(c_1) = e^{-1/2} - 1/4 = \frac{1}{\sqrt{e}} - \frac{1}{4} > 0$

$\Rightarrow f(a_1)\cdot f(c_1) < 0$

- so $a_2 = a_1$, $b_2 = c_1$

$c_2 = \frac{-1+(-\frac{1}{2})}{2} = -3/4$, $f(c_2) = e^{-3/4} - 9/16 < 0$

$\Rightarrow f(b_2)\cdot f(c_2) < 0$

- so $a_3 = c_2 = -\frac{3}{4}$, $b_3 = b_2 = -\frac{1}{2}$ $\Rightarrow c_3 = \frac{-3/4 - 1/2}{2} = -5/8$

$f(-5/8) = e^{-5/8} - \frac{25}{64} > 0$ so $c_4 = \ldots$

Obviously, computers are better suited for this

Bisection algorithm: (pseudocode)

input: a, b (define the interval), M (max number of iterations), δ (stop if error $< \delta$), ε (stop if $f(c) < \varepsilon$)

initialization: $u \leftarrow f(a)$

$v \leftarrow f(b)$

$e \leftarrow b - a$

IF $\text{sign}(u) = \text{sign}(v)$ THEN stop

FOR $k = 1$ to M DO

$e \leftarrow e/2$ (update the error)

$c \leftarrow a + e$ ($a + e = a + \frac{b-a}{2} = \frac{b+a}{2}$)

$w \leftarrow f(c)$ (compute the f.t. value at midpt)

OUTPUT k, c, w, e

IF $|e| < \delta$ or $|w| < \varepsilon$ THEN stop

IF $\text{sign}(w) \neq \text{sign}(u)$ THEN

$b \leftarrow c$ (update the interval & the f.t. value)

$v \leftarrow w$

ELSE

$a \leftarrow c$

$u \leftarrow w$

ENDIF

END DO

Error analysis:

Theorem: If $[a_0, b_0], [a_1, b_1], \ldots [a_n, b_n], \ldots$ denote the intervals that arise in the bisection method then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r \quad \text{s.t. } f(r) = 0$$

Moreover, $r = \lim_{n\to\infty} c_n$ where $c_n = \frac{a_n + b_n}{2}$

and $|r - c_n| \leq 2^{-(n+1)}(b_0 - a_0)$

Proof: Note that either $a_1 = a_0$ & $b_1 \leq b_0$

or $a_1 \geq a_0$ & $b_1 = b_0$

& either way $\begin{cases} b_1 - a_1 = \frac{b_0 - a_0}{2} \\ a_0 \leq a_1 \ \& \ b_0 \geq b_1 \end{cases}$

continuing this way

$$\begin{cases} a_0 \leq a_1 \leq \ldots \leq b_0 \\ b_0 \geq b_1 \geq \ldots \geq a_0 \\ b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) \end{cases}$$

So (a_n) is a non-decreasing seq. with upper bound b_0

$\Rightarrow$ it converges

Similarly (b_n) converges

Moreover $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \frac{1}{2}\cdot\frac{1}{2}(b_{n-2} - a_{n-2})$

$= \cdots = \frac{1}{2^n}(b_0 - a_0)$

$\Rightarrow \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0$

But $f(a_n)f(b_n) \leq 0$ so taking limits

$(f(r))^2 \leq 0 \Rightarrow f(r) = 0$ (r is a root)

Finally $|r - c_n| \leq \frac{b_n - a_n}{2} \leq 2^{-(n+1)}(b_0 - a_0)$ ▨

(r is in $[a_n, b_n]$ & c_n is the midpt.)

Example If $a_0 = 10$ & $b_0 = 20$

How many steps of the bisection method ensure that the relative accuracy is 10^{-7}.

Relative accuracy $\leq 10^{-7} \Rightarrow \frac{|r - c_n|}{|r|} \leq 10^{-7}$

but $r \geq a_0 \geq 10$ so it's enough to have

$|r - c_n| \leq 10^{-8}$

we know that $|r - c_n| \leq 2^{-(n+1)}(20 - 10)$

So we want $10 \cdot 2^{-(n+1)} \leq 10^{-8}$

$-(n+1) \leq \log_2 10^{-9} \Leftrightarrow n \geq (\log_2 10^9) - 1$ so $n \geq 29$

HW 3.1 : 2,4
Computer problems 1,3